Lecture 5

Mathematical Models based on high order differencial equations (Review).

We have a very simple explanation why models based on the second order derivatives (with respect to time) are so popular – such models follow from the Newton laws.

The second Newton's law

$$\begin{cases} \frac{d}{dt}(m v) = F, \\ v = \frac{du}{dt}. \end{cases}$$

This problem has a unique solution if we add <u>two</u> initial conditions

$$\begin{cases} u(0) = u_0, \\ \dfrac{du}{dt}(0) = v_0 \end{cases}$$

If m is constant then we get the second order diff. equation

$$\frac{d}{dt}\left(m\frac{du}{dt}\right) = m\frac{d^2u}{dt^2}$$

$$\Rightarrow \begin{cases} m\dfrac{d^2u}{dt^2} = F, & \text{equation} \\ u(0) = u_0 & \text{initial condition} \\ \dfrac{du}{dt}(0) = v_0. & \end{cases}$$

Example 1. The motion of a body attached ~~to~~ to a spring

- the mass of a body is equal to m
- this body moves without friction
- the spring is deformed from the equilibrium by x length (displacement is equal to x)

The restoring force from the spring is described by Hooke's law

$$F(x,t) = -kx$$ (the linear approximation according the Taylor series)

- k is a spring constant

Such motion is called

Simple Harmonic Motion

It describes a repetitive oscillation

Mathematical Model:

$$\begin{cases} m \dfrac{d^2 x}{dt^2} = -kx, \\ x(0) = x_0, \\ v(0) = V_0 \end{cases}$$

- Friction forces

$$F_{fr} = -\eta \frac{dx}{dt}$$

are proportional ~~to~~ the velocity and arise only if the body is moving (again a linear approx.).

If we consider a math model for the velocity due ~~to~~ the friction forces we get:

$$m \frac{dv}{dt} = -\eta v \, , \quad v(0) = V_0 .$$

Also we should take into account the fact that a mechanical system can be affected by external forces

$f(t)$

Even if $f(t)$ is small at all time moments over a longer period of time this can significantly affect the dynamics of the system (resonance phenomenon).

Now we consider important cases of the model.

1 Free oscillations without a friction

$$m\frac{d^2x}{dt^2}+kx=0.$$

We can normalize this model by using only one parameter a:

$$\frac{d^2x}{dt^2} + a^2x = 0, \quad a^2 = \frac{k}{m}.$$

(Calculate [a])

The general solution is given in the following form

$$x(t) = e^{\lambda t}.$$

Parameter λ satisfies the quadratic equation

$$\lambda^2 + a^2 = 0 \Rightarrow \boxed{\lambda_{1,2} = \pm a i}$$

$$x(t) = c_1 \cos(at) + c_2 \sin(at).$$

$$\omega := a = \sqrt{\frac{k}{m}}$$ (the angular frequency)

Next we use initial conditions

$$x(0) = x_0 \,, \left(\frac{dx}{dt}\right)(0) = 0. \qquad (*)$$

It is clear that initial conditions can be specified differently, depending on a specific experimental setups:

$$x(0) = 0, \left(\frac{dx}{dt}\right)(0) = V_0 \,.$$

A simple task is to find free constants c_1 and c_2 : for IC $(*)$

$$\frac{dx}{dt} = a\left(-c_1 \sin(at) + c_2 \cos(at)\right)$$

$$\left(\frac{dx}{dt}\right)(0) = 0 \Rightarrow \boxed{c_2 = 0}$$

$$x(0) = c_1 \Rightarrow \qquad x(t) = x_0 \cos(at).$$

The solution follows the sinusoid oscillations

Let us compute the velocity of the motion:

$$\frac{dx}{dt} = - x_0 a \sin(at).$$

The velocity ~~is~~ also periodically oscillates.

The period T is defined as

$$aT = 2\pi \Rightarrow T = \frac{2\pi}{a}.$$

Task 1. Find the position $x(t_1)$ and velocity $v(t_1)$ at $t_1 = \frac{\pi}{2a}$.

Task 2 Take initial conditions

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0$$

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t),$$

$$\omega = \sqrt{\frac{k}{m}}.$$

It follows from the IC that

$$x(0) = c_1 \implies c_1 = x_0$$

$$\frac{dx}{dt}(0) = \omega c_2 \implies c_2 = \frac{v_0}{\omega}$$

(initial speed of the particle is divided by the angular frequency).

The solution can be written in the form

$$x(t) = A\cos(\omega t - \varphi)$$

- $A = \sqrt{c_1^2 + c_2^2}$
- $\tan\varphi = c_2 / c_1$

$$\sin\varphi = \frac{c_2}{A}, \quad \cos\varphi = \frac{c_1}{A}.$$

A, ω and φ carries a physical meaning

- A is the amplitude (maximum displacement from the equilibrium position)
- ω is angular frequency
- φ is the initial phase.

$$v(t) = \frac{dx}{dt} = -A\omega \sin(\omega t - \varphi)$$

Find maximum speed, at what point it is reached?

$$a(t) := \frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t - \varphi)$$

(at what points it is reached?)

Now we add the friction term

(η - eta)

$$m \frac{dx^2}{dt^2} + \eta \frac{dx}{dt} + kx = 0.$$

Th object of mass m ~~ex~~periences not only restoring force (whose magnitude is directly proportional to the distance from an equilibrium position) but also the friction force (whose magnitude is directly proportional to the velocity value)

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + a^2 x = 0 \qquad \left(a = \omega = \sqrt{\frac{k}{m}} \right)$$

$$a^2 = \frac{k}{m}, \quad b = \frac{\eta}{m} \qquad [b] \;??$$

A general solution is a linear combination of functions

$$x(t) = e^{\lambda t}.$$

We get the characteristic equation

$$\lambda^2 + b\lambda + a^2 = 0$$

$$\Rightarrow \quad \lambda_{1,2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}$$

Now we investigate <u>important</u> cases

1. $\underline{c^2 := b^2 - 4a^2 \geq 0}$

$$b^2 - 4a^2 \geq 0 \Rightarrow b > 2a$$

$$\Rightarrow \frac{\eta}{m} > 2\sqrt{\frac{k}{m}} \Rightarrow \boxed{\eta > \sqrt{4km}}$$

A general solution of MM is defined as

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Since $\lambda_1, \lambda_2 < 0$, then $x(t) \xrightarrow[t\to\infty]{} 0$

without any oscillations.

The friction is sufficiently strong to exclude a possibility of oscillations

Task. Find c_1 and c_2 (a particular solution~~s~~ satisfy initial conditions)

$$x(0) = x_0, \quad \dot{x}(0) = V_0 \quad \left(\dot{x}(t) = \frac{dx}{dt} \right)$$

Next we consider a case of weaker friction

2. $b^2 - 4a^2 = -c^2 < 0$

$$\boxed{\lambda_{1,2} = -\frac{b}{2} \pm i\,\frac{c}{2}}$$

We get a general solution

$$x(t) = e^{-\frac{b}{2}t}\left[c_1 \cos\left(\frac{c}{2}t\right) + c_2 \sin\left(\frac{c}{2}t\right)\right]$$

The dynamics of this solution is the same (asymptotic value)

$$x(t) \to 0 \quad \text{for} \quad t \to \infty$$

But solution oscillates while its amplitude is decreasing.

Task Make a full analysis of the case $\boxed{b^2 = 4a^2}$

Show a figure describing the dynamics of the solution.

Next we consider the forced oscillations of a system when it is moved out of equilibrium by an external force $f(t)$.

$$\begin{cases} m\dfrac{d^2x}{dt^2} + \eta\dfrac{dx}{dt} + kx = \underbrace{\boxed{A\sin(\omega t)}}_{f(t)} \\ x(0) = 0\,, \quad \dot{x}(0) = 0 \end{cases}$$

Case 1 $\boxed{\text{No friction } \eta = 0}$

- The natural frequency of oscillations does not coincide with the forced frequency of oscillations

$$a \neq \omega \; , \quad a^2 = \frac{k}{m} .$$

We find the solution in the form

$$x(t) = x_1(t) + x_2(t)$$

$$x_1(t) = c_1 \cos(at) + c_2 \sin(at)$$

$$x_2(t) = B \sin(\omega t)$$ (the term due to the force).

We put $x_2(t)$ into equation of MM

$$(-m\omega^2 + k)B = A$$

$$\Rightarrow \quad B = \frac{A}{k - m\omega^2} = \underbrace{\frac{A/m}{a^2 - \omega^2}}$$

The difference of two frequencies.

The solution can be written as

$$\boxed{x(t) = c_1 \cos(at) + c_2 \sin(at) + \frac{A/m}{a^2-\omega^2} \sin \omega t}$$

(dynamics described as a superposition of two oscillation processes)

IC influence

$$x(0) = 0 \Rightarrow \boxed{c_1 = 0}$$

$$\dot{x}(0) := a c_2 + \frac{A/m}{a^2-\omega^2}\omega = 0.$$

We can write the solution in this compact form

$$x(t) = \frac{A/m}{a^2-\omega^2}\left[-\frac{\omega}{a}\sin(at)+\sin(\omega t)\right].$$

Let us consider a limit case

$$\omega \to a$$

We are interested ~~to~~ analyze the dynamics of the spring system when

$$|\omega - a| << 1 \quad (\text{small})$$

L'Hôpital's rule helps to find

$$\lim_{\omega \to a} \frac{-\frac{\omega}{a}\sin(at)+\sin(\omega t)}{a^2-\omega^2}$$

$$= \left.\frac{-\frac{\sin(at)}{a}+t\cos\omega t}{-2\omega}\right|_{\omega=a}$$

$$\xrightarrow[\omega \to a]{} \frac{t\cos(at) - \frac{\sin(at)}{a}}{-2a} \simeq -\frac{t}{2a}\cos(at) \underset{t\to\infty}{\to} \infty$$

Resonance

Resonance is a phenomenon that occurs when an object is subjected to an external force whose frequency matches a resonant frequency of the system.

We are interested in a case, when $\omega \neq a$, but $|\omega - a| << 1$ (small)

$$x(t) = \frac{A/m}{a^2 - \omega^2}\left[-\frac{\omega}{a}\sin(at) + \sin(\omega t)\right]$$

$$= \sin(\omega t) - \sin(at) + \left(1 - \frac{\omega}{a}\right)\sin(at)$$

$$= 2\sin\left(\frac{\omega-a}{2}t\right)\cos\left(\frac{\omega+a}{2}t\right)$$

$$+\left(1-\frac{\omega}{a}\right)\sin(at)$$

The main part of the solution is defined as

$$\tilde{x}(t) = \underbrace{\frac{A/m}{a^2-\omega^2}\cdot 2\cdot\sin\left(\frac{\omega-a}{2}t\right)} \times \underbrace{\cos\left(\frac{\omega+a}{2}t\right)}$$

$M(t)$ - slow harmonic oscillation of the amplitude

Fast oscillations

Task 2. Make analysis of the case

$$\omega = a, \quad 0 < \eta < \sqrt{2km}$$

Mechanical resonance.